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ON ALMOST (ω)REGULAR SPACES

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Abstract. The notion of almost (ω)regular spaces is introduced and some characterizations of the notion are obtained.

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1. Introduction

A set X equipped with a countable number of topologies $\{\mathfrak{J}_n\}$ satisfying $\mathfrak{J}_n \subset \mathfrak{J}_{n+1}, n \in N$ is called an (ω)topological space ([1], [2]) and is denoted by $(X, \{\mathfrak{J}_n\})$. In [3] we provide the motivation for studying such spaces along with that we study the notions of connectedness and hyperconnectedness in an (ω)topological space, while in [5] we deal with compactness and paracompactness in a product (ω)topology. The notion of almost regular topological spaces was introduced by Singal and Arya [4]. In this paper we introduce the notion of almost regularity in (ω)topological spaces. We call it almost (ω)regularity. A set of characterizations of almost (ω)regularity is obtained (Theorem 3.8). Among other results, we prove that the product of arbitrary collection of (ω)topological spaces is almost (ω)regular iff each factor space is almost (ω)regular (Theorem 3.11).

2. Preliminaries

The set of natural numbers and the set of real numbers are denoted by N and R respectively. The elements of N are denoted by i, j, k, l, m, n etc. The closure (resp. interior) of a set $A \subset X$ with respect to a topology $\mathbf{J}$ on X is denoted by $(\mathbf{J})clA$ (resp. $(\mathbf{J})intA$) and the subspace topology of $\mathbf{J}$ on A is denoted by $\mathbf{J}|A$.

Throughout the paper, unless mentioned otherwise, X denotes the (ω)topological space $(X, \{\mathbf{J}_n\})$. For ready reference we give here the following definitions from Bose and Tiwari [1].

Definition 2.1. A set $G(\subset X) \in \mathbf{J}_n$ for some n is called an (ω)*open* set. A set F is said to be (ω)*closed* if $X - F$ is (ω)open. A set $A \subset X$ is said to be (ω)*dense* in X if for every nonempty (ω)open set G, $A \cap G \neq \emptyset$.

Definition 2.2. X is said to be (ω)*Hausdorff* if for any two distinct points x, y of X, there exists an n such that for some $U, V \in \mathbf{J}_n$, we have $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 2.3. X is said to be (ω)*regular* if given an (ω)closed set F and a point $x \in X$ with $x \notin F$, there exists an n such that for some $U, V \in \mathbf{J}_n$, we have $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.

Definition 2.4. X is said to be (ω)*compact* if every (ω)open cover of X has a finite subcover.

3. Almost (ω)regular spaces

We introduce the following definitions.

Definition 3.1. If an (ω)open set G is regularly $(\mathbf{J}_n)$open for some n, then it is said to be a *regularly* (ω)*open* set. G is said to be *strongly regularly* (ω)*open* if it is regularly $(\mathbf{J}_n)$open for all n. An (ω)closed set F is said to be *regularly* (ω)*closed* if it is regularly $(\mathbf{J}_n)$closed for some n. F is said to be *strongly regularly* (ω)*closed* if it is regularly $(\mathbf{J}_n)$closed for all n.

Obviously a set A is regularly (ω)closed iff it is the complement of a regularly (ω)open set.

Definition 3.2. X is said to be *almost* (ω)*regular* if for any regularly (ω)closed set F and any point x with $x \notin F$, there exists an n such that for some $U, V \in \mathbf{J}_n$, we have $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.

Example 3.3. Let us consider the (ω)topological space $(X, \{\mathbf{J}_n\})$ where $X = N$ and the topological space $\mathbf{J}_n$ is defined as follows:

$$\mathbf{J}_1 = \{\emptyset, N, \{1\}\}$$

$$\mathbf{J}_2 = \{\emptyset, N, \{1\}, \{1,2\}\} \text{ and in general,}$$

$$\mathfrak{J}_n = \{\emptyset, N, \{1\}, \{1,2\}, \{1,2,3\}, ..., \{1,2,...,n\}\} \text{ for all } n.$$

Then we see that this (ω)topological space X is almost (ω) regular, however, it is not (ω)regular.

Definition 3.4. X is said to be *semi-*(ω)*regular* if for any $x \in X$ and any (ω)open set G with $x \in G$, there exists an n such that for some regularly $(\mathfrak{J}_n)$open set H, we have $x \in H \subset G$.

Example 3.5. (Bose and Tiwari [1]) Let us consider the increasing sequence $\{\mathfrak{J}_n\}$ of topologies on N defined by $\mathfrak{J}_n = \{N\} \cup P\{1, 2, 3, ..., n\}$, where $P\{1, 2, 3, ..., n\}$ denotes the power set of the set $\{1, 2, 3, ..., n\}$. Then the (ω)topological space $(N, \{\mathfrak{J}_n\})$ is semi-(ω)regular but not (ω)regular.

Definition 3.6. X is said to be an (ω)*Urysohn* space if for any two distinct points $x, y \in X$, there exists an n such that for some $U, V \in \mathfrak{J}_n$, we have $x \in U$, $y \in V$ and $(\mathfrak{J}_n)clU \cap (\mathfrak{J}_n)clV = \emptyset$.

The (ω)topological space in Example 3.5 is (ω)Hausdorff but not (ω)Urysohn.

Definition 3.7. X is said to be *almost* (ω)*compact* if every (ω)open cover $\mathrm{U} = \{U_\alpha | \alpha \in A\}$ of X has a finite subcollection $\mathrm{U}_0 = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_m}\}$ with $\bigcup_{k=1}^{m}((\mathfrak{J}_{n_0})\, clU_{\alpha_k}) = X$, where n_0 is any natural number such that $\mathrm{U}_0 \subset \mathfrak{J}_{n_0}$.

Example 3.5 gives us an (ω)topological space which is almost (ω)compact but not (ω)compact.

Theorem 3.8. *For the* (ω)*topological space X, the following statements are equivalent.*

(i) *X is almost (ω)regular.*

(ii) *For any point $x \in X$ and any regularly (ω)open set H with $x \in H$, there exists an n such that for some $(\mathfrak{J}_n)$open set G, we have $x \in G \subset (\mathfrak{J}_n)clG \subset H$.*

(iii) *For any point $x \in X$ and any regularly (ω)open set H with $x \in H$, there exists an n such that for some regularly $(\mathfrak{J}_n)$open set G, we have $x \in G \subset (\mathfrak{J}_n)clG \subset H$.*

(iv) *For any point $x \in X$ and any (ω)open set H with $x \in H \in \mathfrak{J}_m$, there exists an n such that for some regularly $(\mathfrak{J}_n)$open set G, we have $x \in G \subset (\mathfrak{J}_n)clG \subset (\mathfrak{J}_m)int((\mathfrak{J}_m)clH)$.*

(v) *For any point $x \in X$ and any (ω)open set H with $x \in H \in \mathfrak{J}_m$, there exists an n such that for some $(\mathfrak{J}_n)$open set G, we have $x \in G \subset (\mathfrak{J}_n)clG \subset (\mathfrak{J}_m)int((\mathfrak{J}_m)clH)$.*

(vi) *For any point* $x \in X$ *and regularly* (ω)*closed set* F *with* $x \notin F$*, there exists an* n *such that for some* $G, H \in \mathbf{J}_n$*, we have* $x \in G$*,* $F \subset H$ *and* $(\mathbf{J}_n)clG \cap (\mathbf{J}_n)clH = \emptyset$*.*

(vii) *Every regularly* (ω)*closed set* F *is expressible as the intersection of some regularly* (ω)*closed neighbourhoods of* F*.*

(viii) *Every regularly* (ω)*closed set* F *is identical with the intersection of all* (ω)*closed neighbourhoods of* F*.*

(ix) *For any set* $A \subset X$ *and any regularly* (ω)*open set* G *with* $A \cap G \neq \emptyset$*, there exists an* n *such that for some* $(\mathbf{J}_n)$*open set* H*, we have* $A \cap H \neq \emptyset (\mathbf{J}_n)clH \subset G$*.*

(x) *For any nonempty set* $A \subset X$ *and any regularly* (ω)*closed set* F *with* $A \cap F = \emptyset$*, there exists an* n *such that for some* $(\mathbf{J}_n)$*open sets* G *and* H *we have* $A \cap G \neq \emptyset$*,* $F \subset H$ *and* $G \cap H = \emptyset$*.*

Proof. (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Let H be a regularly (ω)open set with $x \in H$. By (ii), there exists an n such that for some $(\mathbf{J}_n)$open set B, we have $x \in B \subset (\mathbf{J}_n)clB \subset H$. We put $G = (\mathbf{J}_n)int((\mathbf{J}_n)clB)$. But $(\mathbf{J}_n)cl((\mathbf{J}_n)int((\mathbf{J}_n)clB)) = (\mathbf{J}_n)clB$. Therefore, $x \in G \subset (\mathbf{J}_n)clG \subset H$ and G is regularly $(\mathbf{J}_n)$open.

(iii) $\Rightarrow$ (iv): Let H be an (ω)open set with $x \in H \in \mathbf{J}_m$. If $B = (\mathbf{J}_m)int((\mathbf{J}_m)clH)$, then B is a regularly $(\mathbf{J}_m)$open set with $x \in B$. Therefore by (iii), there exists an n such that for some $(\mathbf{J}_n)$open set G, we have

$$x \in G \subset (\mathbf{J}_n)clG \subset B = (\mathbf{J}_m)int((\mathbf{J}_m)clH).$$

(iv) $\Rightarrow$ (v): Obvious.

(v) $\Rightarrow$ (vi): Let F be a regularly (ω)closed set F with $x \notin F$ and let $X - F \in \mathbf{J}_m$. Then $B = X - F$ is a regularly (ω)open set with $x \in B$ and $B \in \mathbf{J}_m$. By (v), there exists a p such that for some $(\mathbf{J}_p)$open set D,

$$x \in D \subset (\mathbf{J}_p)clD \subset (\mathbf{J}_m)int((\mathbf{J}_m)clB) = B.$$

If $A = (\mathbf{J}_p)int((\mathbf{J}_p)clD)$, then A is a regularly $(\mathbf{J}_p)$open set such that $x \in A \subset (\mathbf{J}_p)clA \subset B$. Again by (v), there exists an l such that for some $(\mathbf{J}_l)$open set G, we have $x \in G \subset (\mathbf{J}_l)clG \subset A$. If $H = X - (\mathbf{J}_p)clA$, then $H \in \mathbf{J}_p$ and $F \subset H$.

Now

$$\begin{aligned}(\mathbf{J}_p)clH &= (\mathbf{J}_p)cl(X - (\mathbf{J}_p)clA) \\ &\subset (\mathbf{J}_p)cl(X - A). \\ &= X - A.\end{aligned}$$

Again $(\mathfrak{J}_l)clG \subset A$. Therefore, $(\mathfrak{J}_l)clG \cap (\mathfrak{J}_p)clH = \emptyset$. Let $n = \max(l, p)$.

Then $G, H \in \mathfrak{J}_n$, $x \in G$, $F \subset H$ and $(\mathfrak{J}_n)clG \cap (\mathfrak{J}_n)clH = \emptyset$.

(vi) $\Rightarrow$ (vii): Let F be a regularly (ω)closed set and $x \notin F$. By (vi), there exists an $n_x \in N$ such that for some G_x, $H_x \in \mathfrak{J}_{n_x}$, we have $x \in G_x$, $F \subset H_x$ and $(\mathfrak{J}_{n_x})clG_x \cap (\mathfrak{J}_{n_x})clH_x = \emptyset$. It then follows that $F = \cap_{x \notin F} (\mathfrak{J}_{n_x})cl\, H_x$. Also each $(\mathfrak{J}_{n_x})cl\, H_x$ is a regularly (ω)closed neighbourhood of F.

(vii) $\Rightarrow$ (viii): Obvious.

(viii) $\Rightarrow$ (ix): Let A be a nonempty set and G be a regularly (ω)open set with $A \cap G \neq \emptyset$. $X - G$ is a regularly (ω)closed set. Therefore by (viii),

$$X - G = \bigcap_{\gamma \in \Gamma} M_\gamma,$$

here each M_γ is an (ω)closed neighbourhood of $X - G$. Let $x \in A \cap G$. Then $x \notin X\text{-}G$, so there exists a γ such that $x \notin M_\gamma$. Again there exists an l such that for some $(\mathfrak{J}_l)$open set D, we have $X - G \subset D \subset M_\gamma$. The set $H = X - M_\gamma$ is $(\mathfrak{J}_m)$open for some m and $x \in H$. Let $n = \max(l, m)$. Then $(\mathfrak{J}_n)clH = (\mathfrak{J}_n)cl(X - M_\gamma) \subset (\mathfrak{J}_n)cl(X - D) \subset (\mathfrak{J}_l)cl(X - D) = X - D \subset G$. Also H is $(\mathfrak{J}_n)$open and $A \cap H \neq \emptyset$.

(ix) $\Rightarrow$(x): Let A be a nonempty set and F be a regularly (ω)closed set with $A \cap F = \emptyset$. The set $B = X - F$ is regularly (ω)open with $A \cap B \neq \emptyset$, then there exists an n such that for some $(\mathfrak{J}_n)$open set G with $A \cap G \neq \emptyset$, we have $(\mathfrak{J}_n)clG \subset B$. The set $H = X - (\mathfrak{J}_n)clG$ is $(\mathfrak{J}_n)$open with $F = X - B \subset H$ and $G \cap H = \emptyset$.

(x)$\Rightarrow$ (i): Let F be a regularly (ω)closed set and $x \notin F$. Then we have $\{x\} \cap F = \emptyset$. Therefore by (x), there exists an n such that for some $G, H \in \mathfrak{J}_n$, $\{x\} \cap G \neq \emptyset$, i.e $x \in G$, $F \subset H$ and $G \cap H = \emptyset$. Hence X is almost (ω)regular. □

Theorem 3.9. *Any (ω)dense subspace Y of an almost (ω)regular space X is almost (ω)regular.*

Proof. For any subset E of Y and for any n, we have

$$(\mathfrak{J}_n|Y)int((\mathfrak{J}_n|Y)clE) = (\mathfrak{J}_n)int((\mathfrak{J}_n)clE) \cap Y. \qquad (1)$$

We consider a regularly (ω)open subset H of $(Y, \{\mathfrak{J}_n|Y\})$ with $y \in H$. Then for some m, $(\mathfrak{J}_m|Y)int((\mathfrak{J}_m|Y)clH) = H$. Therefore by (1), $H = (\mathfrak{J}_m)int((\mathfrak{J}_m)clH) \cap Y$. But $(\mathfrak{J}_m)int((\mathfrak{J}_m)clH)$ is a regularly $(\mathfrak{J}_m)$open subset of X. By the almost (ω)regularity of X, there exists, for some l, a $(\mathfrak{J}_l)$open set G such that $y \in G \subset (\mathfrak{J}_l)clG \subset$

H. So $y \in G \subset (\mathbf{J}_l|Y)clG \subset H$. Again since G is $(\mathbf{J}_l)$open, it is $(\mathbf{J}_l|Y$)open. Hence Y is almost (ω)regular. □

Theorem 3.10. *If Y is a strongly regularly (ω)open subspace of an almost (ω)regular space X, then Y is almost (ω)regular.*

Proof. Let H be a regularly (ω)open subset of $(Y, \{\mathbf{J}_n|Y\})$ with $y \in H$.

Then for some m, we have

$$\begin{aligned} H &= (\mathbf{J}_m|Y)int((\mathbf{J}_m|Y)clH) \\ &\supset (\mathbf{J}_m)int((\mathbf{J}_m)clH) \cap (\mathbf{J}_m)intY. \end{aligned} \qquad (2)$$

Since $H \subset Y$ and since Y is strongly regularly (ω)open, we have

$$(\mathbf{J}_m)int((\mathbf{J}_m)clH) \subset (\mathbf{J}_m)int((\mathbf{J}_m)clY) = Y$$
$$\Rightarrow H \supset (\mathbf{J}_m)int((\mathbf{J}_m)clH) \quad \text{(by (2))}$$
$$\Rightarrow H = (\mathbf{J}_m)int((\mathbf{J}_m)clH).$$

Hence H is a regularly (ω)open subset of X with $y \in H$. Since X is almost (ω)regular, there exists an n such that for some $(\mathbf{J}_n)$open set G, we have $y \in G \subset (\mathbf{J}_n)clG \subset H$. Then G is $(\mathbf{J}_n|Y)$open. Thus it follows that $y \in G \subset (\mathbf{J}_n|Y)clG \subset H$. Hence Y is almost (ω)regular. □

Let $\{(X_\alpha, \{\mathbf{J}_n^\alpha\})|\alpha \in A\}$ be a family of (ω)topological spaces and for each n, let $\boldsymbol{P}_n$ be the product topology on $\prod_{\alpha\in A} X_\alpha$ of the product topologies $\mathbf{J}_n^\alpha$, $\alpha \in A$. Then $(\prod_{\alpha\in A} X_\alpha, \{\boldsymbol{P}_n\})$ is called the product (ω)topological space of the spaces $\{(X_\alpha, \{\mathbf{J}_n^\alpha\}), \alpha \in A$.

Theorem 3.11. *The product space $(\prod_{\alpha\in A} X_\alpha, \{\boldsymbol{P}_n\})$ is almost (ω)regular iff each of the factor spaces $(X_\alpha, \{\mathbf{J}_n^\alpha\})$ are almost (ω)regular.*

Proof. Firstly, we suppose each $(X_\alpha, \{\mathbf{J}_n^\alpha\})$ is almost (ω)regular. Let B be any Let B be any (ω)open subset of $(X, \{\boldsymbol{P}_n\})$ where $X = \prod_{\alpha\in A} X_\alpha$ and let B contain a point $x = (x_\alpha) \in X$. If $B \in \boldsymbol{P}_l$, then there exists a set $D \in \boldsymbol{P}_l$, with the following properties: (i) D is of the form $\prod D_\alpha$ where $D_\alpha = X_\alpha$ for all α except for a finite number of indices $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $D_{\alpha_i} \in \mathbf{J}_l^{\alpha_i}$ for $i = 1,2,\ldots,k$, (ii) $x \in D \subset B$. Then $x_{\alpha_i} \in D_{\alpha_i}$. Since each $\left(X_{\alpha_i}, \{\mathbf{J}_n^{\alpha_i}\}\right)$ is almost (ω)regular, by Theorem **3.8** there exists an $n(\alpha_i) \in N$ such that for some $G_{\alpha_i} \in \mathbf{J}_{n(\alpha_i)}^{\alpha_i}$, we have

$$x_{\alpha_i} \in G_{\alpha_i} \subset \left(\mathbf{J}_{n(\alpha_i)}^{\alpha_i}\right) clG_{\alpha_i} \subset (\mathbf{J}_l^{\alpha_i})int\left((\mathbf{J}_l^{\alpha_i})clD_{\alpha_i}\right).$$

If $n = max(n(\alpha_1), n(\alpha_2), \ldots, n(\alpha_k))$, then $G_{\alpha_i} \in \mathbf{J}_n^{\alpha_i}$, $i = 1,2,\ldots,k$ and from above, we get

$$x_{\alpha_i} \in G_{\alpha_i} \subset (\mathbf{J}_n^{\alpha_i})clG_{\alpha_i} \subset (\mathbf{J}_l^{\alpha_i})int\left((\mathbf{J}_l^{\alpha_i})clD_{\alpha_i}\right).$$

We now write $G_\alpha = X_\alpha$ for $\alpha \neq \alpha_i$ and $G = \prod_{\alpha\in A} G_\alpha$. Then $G \in \boldsymbol{P_n}$ and $x \in G$.

Again,

$$G \subset (\boldsymbol{P_n})clG = \prod_{\alpha\in A} (\mathbf{J}_{\boldsymbol{n}}^{\boldsymbol{\alpha}})clG_\alpha$$

$$\subset \prod_{\alpha\in A} (\mathbf{J}_{\boldsymbol{l}}^{\boldsymbol{\alpha}})int\big((\mathbf{J}_{\boldsymbol{l}}^{\boldsymbol{\alpha}})clD_\alpha\big)$$

$$= (\boldsymbol{P_l})int((\boldsymbol{P_l})cl(\textstyle\prod_{\alpha\in A} D_\alpha))$$

$$= (\boldsymbol{P_l})int\big((\boldsymbol{P_l})clD\big) \subset (\boldsymbol{P_l})int\big((\boldsymbol{P_l})clB\big).$$

Hence the product space is almost (ω)regular.

Conversely, let $X = \prod_{\alpha\in A} X_\alpha$ is almost (ω)regular. We consider one factor space X_{α_0} and a $G_{\alpha_0} \in \mathbf{J}_{\boldsymbol{n}}^{\boldsymbol{\alpha_0}}$ for some n. Suppose G_{α_0} contains the point $x_{\alpha_0} \in X_{\alpha_0}$.

For $\alpha \neq \alpha_0$, we write $G_\alpha = X_\alpha$ and $G = \prod_{\alpha\in A} G_\alpha$. Then $G \in \boldsymbol{P_n}$. We choose $x_\alpha \in X_\alpha(\alpha \neq \alpha_0)$ arbitrarily and write $x = (x_\alpha)$. Since $x \in G$ and X is almost (ω)regular, there exists by Theorem 3.8, an m such that for some $H \in \boldsymbol{P_m}$, we have

$$x \in H \subset (\boldsymbol{P_m})clH \subset (\boldsymbol{P_n})int((\boldsymbol{P_n})cl\, G).$$

We may take H in the form $\prod_{\alpha\in A} H_\alpha$ where $H_\alpha = X_\alpha$ for all α except for a finite number of indices $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $H_{\alpha_i} \in \mathbf{J}_{\boldsymbol{m}}^{\boldsymbol{\alpha_i}}$, for $i = 0,1,\ldots,n$. Then

$$x_{\alpha_0} \in H_{\alpha_0} \subset \big(\mathbf{J}_{\boldsymbol{m}}^{\boldsymbol{\alpha_0}}\big)clH_{\alpha_0} \subset \big(\mathbf{J}_{\boldsymbol{n}}^{\boldsymbol{\alpha_0}}\big)int(\big(\mathbf{J}_{\boldsymbol{n}}^{\boldsymbol{\alpha_0}}\big)clG_{\alpha_0}).$$

Hence $X_{\alpha_0}$is almost (ω)regular. □

Theorem 3.12. *If K is an almost* (ω)*compact subset and F is a regularly* (ω)*closed subset of an almost* (ω)*regular space X with $K \cap F = \emptyset$, then there exists an n such that for some $U, V \in \mathbf{J}_{\boldsymbol{n}}$, we have $K \subset U$, $F \subset V$ and $U \cap V = \emptyset$.*

Proof. Since X is almost (ω)regular, by Theorem **3.8**, for $x \in K$, there exists an $n_x \in N$ such that for some $G_x, H_x \in \mathbf{J}_{\boldsymbol{n_x}}$, we have $x \in G_x$, $F \subset H_x$ and $(\mathbf{J}_{\boldsymbol{n_x}})clG_x \cap (\mathbf{J}_{\boldsymbol{n_x}})clH_x = \emptyset$. Then the collection $\mathbf{G} = \{G_x \cap K \mid x \in K\}$ is an (ω)open cover of $(K, \{\mathbf{J}_{\boldsymbol{n}}|K\})$. Since K is almost (ω)compact, $\mathbf{G}$ has a finite subcollection $\mathbf{G}_1$ such that $K = \cup\{(\mathbf{J}_{\boldsymbol{n_0}}|K)clG | G \in \mathbf{G}_1\}$ where n_0 is any natural number such that $\mathbf{G}_1 \subset \mathbf{J}_{\boldsymbol{n_0}}$. Suppose $G_{x_i} \cap K \in \mathbf{G}_1$, $i = 1\,2, \ldots, k$. Therefore,

$$K \subset \cup_{i=1}^{k} (\mathbf{J}_{\boldsymbol{n_0}})clG_{x_i}$$

$$\subset X - \bigcap_{i=1}^{k} (\mathbf{J}_{n_0}) clH_{x_i} = U(\text{say}).$$

Again $F \subset \bigcap_{i=1}^{k} H_{x_i} = V(\text{say})$. Also $U, V \in \mathbf{J}_{n_0}$ and $U \cap V = \emptyset$. □

Theorem 3.13. *The* (*ω*)*topological space X is* (*ω*)*regular iff it is both almost* (*ω*)*regular and semi-*(*ω*)*regular.*

Proof. We only prove the 'if' part. The other part is obvious.

Let $x \in X$ and G be an (*ω*)open set with $x \in G$. Since X is semi-(*ω*)regular, there exists an n such that for some regularly $(\mathbf{J}_n)$open set H, we have

$$x \in H = (\mathbf{J}_n) int((\mathbf{J}_n) clH) \subset G.$$

Again since X is almost (*ω*)regular, by Theorem **3.8**, there exists an m such that for some $D \in \mathbf{J}_m$ we have,

$$x \in D \subset (\mathbf{J}_m) clD \subset (\mathbf{J}_n) int((\mathbf{J}_n) clH)$$
$$\Rightarrow x \in D \subset (\mathbf{J}_m) clD \subset G.$$

Hence X is (*ω*)regular. □

Theorem 3.14. *If X is almost* (*ω*)*regular and* (*ω*)*Hausdorff, then it is an* (*ω*)*Urysohn space.*

Proof. Let x and y be two distinct points of X. Since X is (*ω*)Hausdorff, there exists an n such that for some $G, H \in \mathbf{J}_n$, we have $x \in G$, $y \in H$ and $G \cap H = \emptyset$. But $G \cap H = \emptyset \Rightarrow (\mathbf{J}_n) clG \cap H = \emptyset$. So $y \notin (\mathbf{J}_n) clG$. Therefore, by almost (*ω*)regularity of X, there exists an m such that for some $D \in \mathbf{J}_m$, we have $y \in D \subset (\mathbf{J}_m) clD \subset X - (\mathbf{J}_n) clG$. So $(\mathbf{J}_n) clG \cap (\mathbf{J}_m) clD = \emptyset$. If $l = \max(m, n)$, then $(\mathbf{J}_l) clG \cap (\mathbf{J}_l) clD = \emptyset$. Also $G, D \in \mathbf{J}_l$ and $x \in G$, $y \in D$. □

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